Episodes in the History of Infinitesimals

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NUMTA 2016, Pizzo Calabro (VV), Italy, june 20, 2016

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infinitesimals were present from the beginning of western mathematics,

but have had and continue to have a contrasted, fluctuating status:

often declared dead, always resurrecting

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spreading of the axiomatic method made obsolete metaphysical disputations

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- the algebraic simplification of nonstandard analysis
  devised by Detlef Laugwitz (1932-2000)

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A. L. Cauchy, *Cours d'analyse de l'Ecole royale polytechnique*, Debures, Paris, 1821

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Cauchy's insistence in reasoning with infinitesimals was also responsible, according to the *vulgata*, for his famous "errors".

[. . .] he [Newton] laid down the Idea of deducing the Area from the Ordinate, by considering the Area as a Quantity, growing or increasing by continual Flux, in Proportion to the Length of the Ordinate, supposing the Abscissa to increase uniformly in Proportion to Time. And from the Moments of Time he gave the Name of Moments to the momentaneous Increases, or infinitely small Parts of the Abscissa and Area, generated in Moments of Time. The Moment of a Line he called a Point, in the Sense of Cavallerius, tho' it be not a geometrical Point, but a Line infinitely short, and the Moment of an Area or Superficies he called a Line, in the Sense of Cavallerius, though it be not a geometrical Line, but a Superficies infinitely narrow.

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Cavallerius is Bonaventura Cavalieri (1598-1647), who called his infinitesimal lines "indivisibles"

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This ratio [$a : y + z$] is always smaller than $a : 2y$, but the smaller z is, the greater the ration will be and, since one may choose z as small as one pleases, the ratio $a : 2y + z$ can be brought as close to the ratio $a : 2y$ as we like. Consequently, $a : 2y$ is the limit of the ratio $a : 2y + z$.

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When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last fixed value is called the limit of all the others.

definition of limit:

> When the numerical successive values [absolute values] of a variable decrease indefinitely in such a way as to become less than any given number, this variable becomes what is called an *infinitesimal* or a quantity *infinitely small*.

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the use of infinitely large numbers is not frequent in Cauchy's proofs there is no definition of pointwise continuity,

but only of global continuity in an interval

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continuity in an interval

> If, starting from a value of x included between these limits, one assigns to the variable x an infinitely small increment α, the function itself will take on for an increment the difference $f(x+\alpha)-f(x)$, which will depend at the same time on the new variable α and on the value of x. This granted, the function $f(x)$ will be, between the two limits assigned to the variable x, a *continuous* function of the variable if, for each value of x intermediate between these limits, the numerical value of the difference $f(x+\alpha)-f(x)$ decreases indefinitely with that of α. In *other words, the function $f(x)$ will remain continuous with respect to x between the given limits, if, between these limits, an infinitely small increment of the variable always produces an infinitely small increment of the function itself*.
>
> Cauchy, 1821

continuity in the vicinity of a particular value:

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> Unfortunately, Cauchy claimed to be able to prove the existence of the integral, that is the convergence of the "Riemann's sums" for an arbitrary continuous function; his proof, that would be correct if based on the theorem of uniform continuity of functions continuous in a closed interval, is deprived of any probative value by want of such a notion.
>
> Bourbaki, 1960

approximating sums over subdivisions of an interval $x_0 \leq x \leq X$

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if $x_0 < x_1 < \ldots < x_{n-1} < X$ is the common refinement of two subdivisions of the interval

the difference of the sums is

$$D = \epsilon_0(x_1 - x_0) + \epsilon_1(x_2 - x_1) + \ldots + \epsilon_{n-1}(X - x_{n-1})$$

where each ϵ_i is the difference of the values of the function

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D will be infinitesimal, by the continuity of the function and

$$D = \epsilon(X - x_0)$$

where ϵ is a mean of the ϵ_i

Therefore, when the elements of the difference $X - x_0$ become infinitely small, the mode of division has no more than an imperceptible [*insensible*] influence on the value of S; and, if one makes the numerical values of these elements decrease indefinitely, by increasing their number, the value of S will end by being perceptibly [*sensiblement*] constant or, in other words, it will end by attaining a certain limit which will depend solely on the form of the function $f(x)$ and on the extreme values x_0 and X attributed to the variable x. This limit is that which one calls a definite integral.

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this hidden lemma is true in any reasonable theory of infinitesimals

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> [...] in the second edition page 9 [of Thomae's book] one finds numbers that (*horribile dictu*) are smaller than any conceivable real number, and yet are different from zero.

letter to Giulio Vivanti (1859-1949) of December 13, 1893:

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> [Differentials simply are variables having zero as their limit, and represent the potentially infinitely small, without being properly infinitesimal. However] this does not exclude that, in a future state of analysis, one may find means to define different quantities, since they would be smaller than any of the quantities used until now; these properly infinitesimal quantities will certainly in no way be related to our differentials.

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Lasswitz follower of Hermann Cohen (1842-1918)

Gösta Mittag-Leffler (1846-1927) in a letter of February 7, 1883 asked Cantor

1. "Are there among your new numbers such that might fit in between the rational and irrational numbers?"

2. "If $\sum a_k$ and $\sum b_k$ are divergent series of positive terms, can one say which one is greater?"

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$1 + 2 + 3 + \ldots. = \omega$, $1 + 1/2 + 1/3 + \ldots = \omega$

$2 + 3 + \ldots + 1 = \omega + 1$

$1 + 3 + 5 + \ldots + 2 + 4 + 6 + \ldots = 2\omega$

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The definition of the sum of a series of positive numbers, which is given as a well-ordered set, is obtained by the least hyperfinite [*überendliche*] number, which is greater than or equal to the sum of arbitrarily many numbers of the set, taken in their given succession: that such a minimum always exists is easily seen.

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$$\frac{1}{\omega}, \frac{1}{\omega+1}, \ldots, \frac{1}{\omega^2}$$

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1. "Addition of any finite collection of these quantities is always possible,

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   finite multiples $\zeta \cdot \nu$ of any quantity ζ are possible, if ν is a finite integral multiplier.

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   finite multiples $\zeta \cdot \nu$ of any quantity ζ are possible, if ν is a finite integral multiplier.

2. Also, a simply infinite sequence $\zeta_1, \zeta_2, \zeta_3, \ldots$ of those

   quantities, taken in the given succession of its summands,

   must have a definite sum s, where s belongs either to the old or to the extended system."

the existence of the least upper bound in a disguised form:

"In order that s be the sum of that infinite series,

if s' is *any* quantity (among those known) smaller than s,

then a finite integer n must exist such that $\zeta_1 + \zeta_2 + \ldots + \zeta_n > s'$.

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Let $s' = 3/4$, from postulate 2 for some finite integer n:

$$\text{(B) } \zeta_1 + \zeta_2 + \ldots + \zeta_n > 3/4.$$

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Now if $\zeta = 1/\omega$, or $\zeta \cdot \omega = 1$, let $\zeta_1 = \zeta_2 = \zeta_3 = \ldots = \zeta$.

Then, (A) $\zeta_1 + \zeta_2 + \ldots \text{in inf.} = 1$.

Let $s' = 3/4$, from postulate 2 for some finite integer n:

(B) $\zeta_1 + \zeta_2 + \ldots + \zeta_n > 3/4$.

Since all $\zeta_k = \zeta$, (B) implies $\zeta_{n+1} + \zeta_{n+a} + \ldots \zeta_{2n} > 3/4$,

and, finally, (C) $\zeta_1 + \zeta_2 + \ldots + \zeta_n + \ldots \zeta_{2n} > 3/4 + 3/4$,

the existence of the least upper bound in a disguised form:

"In order that s be the sum of that infinite series,

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a contradiction to (A).

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> We certainly do not give laws to the intellect or other things by our arbitrary will, but as faithful scribes we receive and copy them from the very voice of nature.
>
> Francis Bacon, 1653

[...] mine [numbers] depend upon *absolutely no hypotheses*, but are immediately derived from the natural concept of sets. They are just as necessary and free from arbitrariness as the finite whole numbers.

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"And so one could continue at length the enumeration of the absurdities which the author has piled up. But these errors, and the lack of precision and rigor throughout the book, deprive it of any value"

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all known examples of infinitesimals were all determined by functions,

“but among the usual magnitudes, e. g. among linear segments, do infinitesimals exist?

Cantor answered in the negative; but the proof given by this eminent mathematician

is so synthetic that it was considered incomplete.

The aim of this note is to expand such proof”.

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∞u is a segment in the following more general sense:

it is not empty; it is different from the whole half line;

any point between the origin and one of its points belong to it;

every one of its points falls between the origin and some other of its points.

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"if u is infinitesimal with respect to v, ∞u is a segment contained in v.

Hence we can add to ∞u the segment u, and obtain the segment $(\infty + 1)u$".

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the sum $u+v$ of two segments is the set of all the right endpoints of the segments

obtained by concatenating a segment whose right endpoint belong to u

with a segment whose right endpoint belongs to v

Peano: if we construct $(\infty+1)u, (\infty+2)u, \ldots, 2\infty u, \ldots, \infty^2 u$

[the last being obtained by multiplying ∞u by ∞]

and so on, "all these segments, obtained by multiplying u

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“Every segment that can be covered by the sum

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As a consequence, although the segment ∞u is contained in the segment v, it cannot be assigned a definite right endpoint, since when to such a segment we add u, or double it, we get a larger segment.

Each of these results contradicts the common idea of segment. And from the circumstance that the infinitesimal segment cannot be made finite through any actually infinite multiplication, however powerful, I am led to agree with Cantor that it cannot be one of the finite magnitudes

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simplified aproaches to nonstandard analysis

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a different algebraic approach of Laugwitz,

inspired by the extension procedures of the theory of ordered fields

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write ${}^*S$ for the object in $T\langle\Omega\rangle$ corresponding

to the (almost everywhere) constant sequence $S(n) = S$

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$$\beta = \alpha(\nu), \;\; \nu \in^* \mathbb{N}$$

means that for some $a(n,m)$ and some sequence $b(n)$ of objects of T

$b(\Omega) = a(\Omega, m(\Omega))$ where $b(\Omega) = \beta$, $m(\Omega) = \nu$, $a(\Omega, m(\Omega)) = \alpha(\nu)$.

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Lemma 5. Let $\alpha(\mu)$ be a standard sequence. All $\alpha(\mu)$ for infinite μ belong to the same finite monad if and only if $\alpha(\mu) \approx \alpha(\lambda)$ for all infinite μ and λ.

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$T\langle\Omega\rangle$ “is equivalent to the theory of some reduced ultrapower ${}^*S$

where S is the basic set of the theory T

Colin MacLaurin (1698-1746)

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